Affinoids in the Lubin-Tate perfectoid space and simple epipelagic representations

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Abstract: We will construct affinoids in the Lubin-Tate perfectoid space, and discuss their relation with the local Langlands correspondence (LLC) and the local Jacquet-Langlands correspondence (LJLC) for representations of exponential Swan conductor 1, which we call simple epipelagic.

Notation: Let K be a non-archimedean local field. Let $\mathfrak{p}$ be the prime ideal of K, and k be the residue field of K. We put $q = |k|$. Let n be a positive INTEGER.

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1 Lubin-Tate perfectoid space

Let $\mathrm{LT}_n(\mathfrak{p}^i)$ be the Lubin-Tate space with level $\mathfrak{p}^i$ for the 1-dimensional formal $\mathcal{O}_K$-module of height n.

The Lubin-Tate perfectoid space $\mathcal{M}$ is a limit of $\mathrm{LT}_n(\mathfrak{p}^i)$ with respect to i in some sense. We can construct $\mathcal{M}$ as the generic fiber of some formal scheme Spf A. The structure of A is given by the following theorem due to Weinstein:

Theorem ([We]). *We have a natural isomorphism*

$$A \simeq \mathcal{O}_{\mathbf{C}}[[X_1^{1/q^\infty}, \dots, X_n^{1/q^\infty}]]/(\delta(X_1, \dots, X_n)^{q^{-m}} - t^{q^{-m}})_{m \geq 0},$$

where we can express $\delta(X_1, \dots, X_n)$ and t explicitly.

Let $\mathbf{C} = \widehat{\overline{K}}$, and D be the central division algebra over K of invariant $1/n$. We put

$$G = GL_n(K) \times D^\times \times W_K.$$

The base change $\mathcal{M}_{\mathbf{C}}$ has an action of

$$G^1 = \{(g, d, \sigma) \in G \mid \det(g) = \mathrm{Nrd}_{D/K}(d)\mathrm{Art}_K^{-1}(\sigma)\}.$$

Boyarchenko-Weinstein constructed a family of affinoid in $\mathcal{M}_{\mathbf{C}}$ based on the above theorem, and relate them with LLC and LJLC for representations of "unramified type" in [BW].

We want to treat the case for "ramified type". Simple epipelagic representations are the representations of "ramified type" with the smallest conductor.

2 Result

Theorem. *We have a family of affinoids $\{\mathcal{X}_i\}_{i \in I}$ in $\mathcal{M}_{\mathbf{C}}$ and their formal models $\{\mathfrak{X}_i\}_{i \in I}$ such that*

- *the special fiber $\overline{\mathfrak{X}_i}$ of $\mathfrak{X}_i$ is isomorphic to the perfection of a smooth Artin-Schreier variety,*
- *the stabilizer $H_i \subset G^1$ of $\mathcal{X}_i$ naturally acts on $\overline{\mathfrak{X}_i}$, and*
- $\mathrm{c\text{-}Ind}_{H_i}^G H_{\mathrm{c}}^{n-1}(\overline{\mathfrak{X}_i}, \overline{\mathbb{Q}}_\ell)$ *realizes the LLC and the LJLC for simple epipelagic representations.*

We will explain how to construct an affinoid in the above family. Let τ be a simple epipelagic representation of W_K.

3 Tame case

Consider the case where $p \nmid n$. Then we have

$$\tau \simeq \mathrm{Ind}_{W_L}^{W_K} \chi,$$

where L is a totally ramified extension of K, and χ is a character of W_L.

We can construct a CM point $\xi_L \in \mathcal{M}_{\mathbf{C}}$ using a 1-dimensional formal $\mathcal{O}_L$-module over $\mathcal{O}_{L^{\mathrm{ur}}}$. We define an affinoid $\mathcal{X}^L \subset \mathcal{M}_{\mathbf{C}}$ by

$$v\left(\frac{X_i}{\xi_{L,i}} - 1\right) \geq \frac{1}{2nq^{i-1}} \quad \text{for } 1 \leq i \leq n.$$

The special fiber of a formal model of $\mathcal{X}^L$ is isomorphic to the perfection of the variety defined by

$$z^q - z = \sum_{1 \leq i \leq j \leq n-1} y_i y_j \quad \text{in } \mathbb{A}^n_{k^{\mathrm{ac}}}.$$

4 Wild case

Consider the case where $p \mid n$. Assume that $n = p^e$ for simplicity. Then τ **can not** be written as an induction of a character. So we have no expectation that the desired affinoid lives near a CM point.

However, there is a tamely ramified extension T/K such that

$$\tau|_{W_T} \simeq \mathrm{Ind}_{W_M}^{W_T} \chi,$$

where χ is a character of W_M. We can describe explicitly a finite Galois extension L/M such that χ factor through $\mathrm{Gal}(L/M)$.

We will modify coordinates of a CM point using the extension L/T. Let ξ be the modified point. Then we can define the desired affinoid using ξ. The definition is similar to , but a little more complicated.

We put $f = log_p q$ and $m = \gcd(e, f)$. Then the special fiber of a formal model of the affinoid is isomorphic to the perfection of the variety defined by

$$z^{p^m} - z = y^{p^e+1} - \sum_{1 \leq i \leq j \leq n-2} y_i y_j \quad \text{in } \mathbb{A}^n_{k^{\mathrm{ac}}}.$$

5 LLC and LJLC

Cohomology of the above Artin-Schreier varieties realize correspondences between representation of $GL_n(K)$, $D^\times$ and W_K. It is a non-trivial problem to show the obtained correspondences are LLC and LJLC. We can show it for LJLC based on the usual characterization by the character relation. For LLC, we use a simple characterization of LLC for simple epipelagic representations in [BH].

References

[BW] M. Boyarchenko and J. Weinstein, *Maximal varieties and the local langlands correspondence for $GL(n)$*, to appear in J. Amer. Math. Soc.

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