Critical values for automorphic L-functions

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Introduction

In [Har97], M. Harris has defined complex invariants, called automorphic periods, for certain automorphic representations of GL_n over quadratic imaginary field. He proved that critical values of automorphic L-functions for $GL_n \times GL_1$ can be interpreted in terms of automorphic periods.
His results have been generalized to the case $GL_n \times GL_{n'}$ recently. Moreover, we have formulated a concise expression for general critical values. Our formula is compatible with Deligne's conjecture (c.f. [Del79]).

Notation and conventions

We fix $\overline{\mathbb{Q}}$ an algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.
Let $K \subset \overline{\mathbb{Q}}$ be a quadratic imaginary field.
Fix n, n' two integers at least 2.
Let Π (resp. Π') be a cuspidal representation of $GL_n(\mathbb{A}_K)$ (resp. $GL_{n'}(\mathbb{A}_K)$) which is regular, cohomological and conjugate self-dual.
For an integer $0 \leq s \leq n$, if Π descends by base change to a unitary group over $\mathbb{Q}$ of infinity sign $(n-s, s)$ then Π can be realized in the coherent cohomology of the Shimura variety associated to the similitude unitary group. The coherent cohomology has a rational structure over a number field $E(\Pi)$. Harris has defined the **automorphic periods** $P^{(s)}(\Pi)$ as a complex number well defined up to $E(\Pi)^\times$. It is defined as the Petersson inner product of a rational element in the coherent cohomology of certain Shimura variety associated to unitary groups.
We assume that Π descends to unitary groups for all infinity signs henceforth. Therefore the automorphic periods can be defined for every $0 \leq s \leq n$. We postulate the similar assumption for Π'.
For two complex numbers x, y and a number field E, we say $x \sim_E y$ if $y \neq 0$ and $x/y \in E^\times$.

Split Index

We write the infinty type of Π and Π' by $(z^{a_i}\overline{z}^{-a_i})_{1\leq i\leq n}$, $a_1 > a_2 > \cdots > a_n$ and $(z^{b_j}\overline{z}^{-b_j})_{1\leq j\leq n'}$, $b_1 > b_2 > \cdots > b_{n'}$ respectively. We assume that $a_i + b_j \neq 0$ for all $1 \leq i \leq n$ and all $1 \leq j \leq n'$.
We split the sequence $(a_1 > a_2 > \cdots > a_n)$ with the numbers $-b_{n'} > -b_{n'-1} > \cdots > -b_1$. This sequence is split into $n'+1$ parts. We denote the length of each part by $sp(0, \Pi'; \Pi), sp(1, \Pi'; \Pi), \cdots, sp(n', \Pi'; \Pi)$ and call them the **split indices**.

An automorphic version of Deligne's conjecture

The following conjecture is formulated in our work recently. It is already verified in several cases.
Conjecture: Let Π and Π' be as above. Let $m \in \mathbb{Z} + \frac{n+n'}{2}$ be critical for $\Pi \otimes \Pi'$. We predict that:

$$L(m, \Pi \times \Pi') \sim_{E(\Pi)E(\Pi')} (2\pi i)^{nn'm} \prod_{j=0}^{n} P^{(j)}(\Pi)^{sp(j,\Pi;\Pi')} \prod_{k=0}^{n'} P^{(k)}(\Pi')^{sp(k,\Pi';\Pi)}.$$

Moreover, this relation is equivariant under the action of $Gal(\overline{\mathbb{Q}}/K)$.

Known cases

Definition: We say the pair (Π, Π') is **in good position** if $n > n'$ and the numbers $-b_{n'} > -b_{n'-1} > \cdots > -b_1$ are in different gaps between $a_1 > a_2 > \cdots > a_n$.
We say Π is **very regular** if $a_i - a_{i+1} \geq 3$ for all $1 \leq i \leq n-1$.

Here is a list of known cases for the above conjecture:

Case 1: $n' = 1$ and $m \geq \dfrac{1}{2}$. It is shown in [Har97].

Case 2: $n > n'$, Π, Π' very regular, in good position and $m > \frac{1}{2}$ or $m = \frac{1}{2}$ along with a non vanishing condition. When $n' = n-1$ this is proved in [GH15] and [LIN15]. For general n' this is in the ongoing thesis of the author.

Case 3: arbitrary n, n' and arbitrary position for very regular (Π, Π') but $m = 1$. This is also in the ongoing thesis of the author.

Remark: The above results can be generalized to arbitrary CM field.

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Motivic approach and Deligne's conjecture

Let $M^\#$ be a motive over $\mathbb{Q}$ with coefficients in a number field E of weight $\omega \in \mathbb{Z}$.
Recall that its Betti realization $M_B^\#$ and de Rham realization $M_{DR}^\#$ are both finite dimensional vector spaces over E where the former is endowed with a Hodge structure and the latter is endowed with a Hodge filtration.

More precisely, we have a decomposition $M_B^\# \otimes \mathbb{C} = \bigoplus_{p,q\in\mathbb{Z}} M^{p,q}$ as $E \otimes \mathbb{C}$-module and a filtration $M_{DR}^\# = \cdots \supset M^i \supset M^{i+1} \supset \cdots$ as E-module. Moreover, there is a comparison isomorphism $I_\infty : M_B^\# \otimes \mathbb{C} \xrightarrow{\sim} M_{DR}^\# \otimes \mathbb{C}$ as $E \otimes_{\mathbb{Q}} \mathbb{C}$-module such that $I_\infty(\bigoplus_{p\geq i} M^{p,q}) = M^i \otimes \mathbb{C}$.

The infinity Frobenius acts on $M_B^\#$ and exchanges $M^{p,q}$ with $M^{q,p}$. We define $(M_B^\#)^+$ to be the subspace of $M_B^\#$ fixed by the infinity Frobenius. For simplicity we assume that $M^\#$ has no $(\omega/2, \omega/2)$-class and define $F^+(M^\#)$ to be $M^{\omega/2}$. It is easy to see that the comparison isomorphism induces an isomorphism

$$(M_B^\#)^+ \otimes \mathbb{C} \hookrightarrow M_B^\# \otimes \mathbb{C} \xrightarrow{\sim} M_{DR}^\# \otimes \mathbb{C} \to (M_{DR}^\#/F^+(M^\#)) \otimes \mathbb{C}.$$

Deligne's period $c^+(M^\#)$ is defined to be the determinant of the above isomorphism with respect to any fixed E-bases of $(M_B^\#)^+$ and $M_{DR}^\#/F^+(M^\#)$. It is well defined up to $E^\times$.
Deligne has predicted in [Del79] that $L(m, M^\#) \sim_E (2\pi i)^{m*dim(M^\#)} c^+(M^\#)$ if m is critical for $M^\#$.

Deligne's period for automorphic pairs

Let M and M' be two regular motives over K of dimension n and n', with coefficients in E and E' respectively. The **motivic periods** $Q_i(M)$ can be defined for $1 \leq i \leq n$ as in [Har13]. It is the ratio of two rational elements respect to two different rational structures in the i-th level of the Hodge decomposition. The **determinant period** $\delta^{Del}(M)$ is defined as the determinant of the comparison isomorphism $I_\infty : M_B \otimes \mathbb{C} \xrightarrow{\sim} M_{DR} \otimes \mathbb{C}$. It is an analogue as the determinant period in [Del79] where the motives are over $\mathbb{Q}$.

Let $M^\#$ be the restriction of $M \otimes M'$ from K to $\mathbb{Q}$. It is a motive over $\mathbb{Q}$. We may calculate Deligne's period $c^+(M^\#)$ explicitly. The right formula should be the inverse of that in Lemma 1.4.1 of [Har13].
An important ingredient of the ongoing thesis of the author is to simplify the expression for $c^+(M)$ when M and M' are associated to automorphic pairs.

Let us assume that there exists motives M and M' associated to Π and Π' respectively. For all $0 \leq j \leq n$ we define the **motivic periods** $Q^{(j)}(M) := Q_1(M)^{-1} \cdots Q_j(M)^{-1} \delta^{Del}(\xi_\Pi)$ where ξ_Π is the central character of Π. We define $Q^{(k)}(M')$ for $1 \leq k \leq n'$ similarly.
The motivic period $Q^{(j)}(M)$ is related to the automorphic period $P^{(j)}(\Pi)$. The comparison is done in section 4 of [GH15].

Proposition: If $M^\#$ has no (p,p)-class then

$$c^+(M^\#) \sim_{E(\Pi)E(\Pi')} (2\pi i)^{\frac{-nn'(n+n'-2)}{2}} \prod_{j=0}^{n} Q^{(j)}(M)^{sp(j,\Pi;\Pi')} \prod_{k=0}^{n'} Q^{(k)}(M')^{sp(k,\Pi';\Pi)}.$$

At last, since $L(m, \Pi \times \Pi') = L(m + \frac{n+n'-2}{2}, M^\#)$, our conjecture is compatible with Deligne's conjecture.

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