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On L-packets and depth for $SL_2(K)$

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Abstract

Let $\mathcal{G} = \mathrm{SL}_2(K)$ with K a local function field of characteristic 2. We review the Artin-Schreier symbol for the field K, and show that this leads to a parametrization of certain L-packets in the smooth dual of $\mathcal{G}$. The L-packets in the principal series are parametrized by quadratic extensions, and the supercuspidal L-packets of cardinality 4 are parametrized by biquadratic extensions. Each supercuspidal packet of cardinality 4 is accompanied by a singleton packet for $\mathrm{SL}_1(D)$. We provide lower bounds for the depths of the irreducible constituents of all these L-packets for $\mathrm{SL}_2(K)$ and its inner form $\mathrm{SL}_1(D)$.

Quadratic characters

Let E/K be the quadratic extension given by

$$E = K(\wp^{-1}(u_j\varpi^{-2n-1})),$$

where $\mathcal{B} = \{u_1, \ldots, u_f\}$ is a basis of the $\mathbb{F}_2$-linear space $\mathbb{F}_q = \mathbb{F}_{2^f}$.
The Artin-Schreier symbol creates a sequence of quadratic characters

$$\chi_{n,j}(\alpha) := (\alpha, u_j\varpi^{-2n-1} + \wp(K)] \qquad (1)$$

with $n \geq 0$ and $j = 1, \ldots, f$.

Theorem (Explicit formula for the Artin-Schreier symbol)

Let K be a local function field of characteristic 2 with residue degree f. Then,

$$\chi_{n,j}(\alpha) = \sum_{i|2n+1} Tr_{\mathbb{F}_q/\mathbb{F}_2}(u_j\theta_i^{(2n+1)/i})$$

where $\alpha = \varpi^k\theta_0\prod_{i\geq 1}(1+\theta_i\varpi^i) \in K^\times$, $n \geq 0$ and $j = 1, \ldots, f$.

Depth

The depth of a Langlands parameter ϕ is defined as follows. Let r be a real number, $r \geq 0$, let $\mathrm{Gal}(K_s/K)^r$ be the r-th ramification subgroup of the absolute Galois group of K. Then the depth of ϕ is the smallest number $d(\phi) \geq 0$ such that ϕ is trivial on $\mathrm{Gal}(K_s/K)^r$ for all $r > d(\phi)$.

The *depth* $d(\pi)$ of an irreducible $\mathcal{G}$-representation π was defined by Moy and Prasad in terms of filtrations $P_{x,r}(r \in \mathbb{R}_{\geq 0})$ of the parahoric subgroups $P_x \subset \mathcal{G}$.

L-packets of cardinality 4

Following [We], denote by α, β, γ the images in $\mathrm{PSL}_2(\mathbb{C})$ of the elements

$$z_\alpha = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad z_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad z_\gamma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

in $\mathrm{SL}_2(\mathbb{C})$. Note that $z_\alpha, z_\beta, z_\gamma \in \mathrm{SU}_2(\mathbb{C})$ so that

$$\alpha, \beta, \gamma \in \mathrm{PSU}_2(\mathbb{C}) = \mathrm{SO}_3(\mathbb{R}).$$

Denote by J the group generated by α, β, γ:

$$J := \{\epsilon, \alpha, \beta, \gamma\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

The group J is unique up to conjugacy in $G = \mathrm{PSL}_2(\mathbb{C})$.
The pre-image of J in $\mathrm{SL}_2(\mathbb{C})$ is the group $\{\pm 1, \pm z_\alpha, \pm z_\beta, \pm z_\gamma\}$ and is isomorphic to the group U_8 of unit quaternions $\{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$.

Let $\mathbf{W}_K$ denote the Weil group of K. Let L/K be a biquadratic extension of K. An identification $\mathrm{Gal}(L/K) \simeq J$, composed with the inclusion $J \to \mathrm{SO}_3(\mathbb{R})$ determines a Langlands parameter as follows:

$$\phi = \phi_L : \mathbf{W}_K \to \mathrm{Gal}(L/K) \to \mathrm{SO}_3(\mathbb{R}) \subset \mathrm{PSL}_2(\mathbb{C}). \qquad (2)$$

Define

$$S_\phi = C_{\mathrm{PSL}_2(\mathbb{C})}(\mathrm{im}\,\phi) \qquad (3)$$

Define the new group

$$\mathcal{S}_\phi = C_{\mathrm{SL}_2(\mathbb{C})}(\mathrm{im}\,\phi)$$

The article [ABPS2] finalizes the local Langlands correspondence for any inner form of SL_n over all local fields.

We have the short exact sequence

$$1 \to \mathcal{Z}_\phi \to \mathcal{S}_\phi \to S_\phi \to 1$$

with $\mathcal{Z}_\phi = \mathbb{Z}/2\mathbb{Z}$.
Let D be a central division algebra of dimension 4 over K, and let Nrd denote the reduced norm on $D^\times$. Define

$$\mathrm{SL}_1(D) = \{x \in D^\times : \mathrm{Nrd}(x) = 1\}.$$

Then $\mathrm{SL}_1(D)$ is an inner form of $\mathrm{SL}_2(K)$. In the local Langlands correspondence [ABPS2] for the inner forms of SL_2, the L-parameter ϕ is enhanced by elements $\rho \in \mathbf{Irr}(\mathcal{S}_\phi)$. Now the group $\mathcal{S}_\phi \simeq U_8$ admits four characters $\rho_1, \rho_2, \rho_3, \rho_4$ and one irreducible representation ρ_0 of degree 2.
The parameter ϕ creates a Vogan packet with five elements, which are allocated to $\mathrm{SL}_2(K)$ or $\mathrm{SL}_1(D)$ according to central characters.

So ϕ assigns an L-packet Π_ϕ to $\mathrm{SL}_2(K)$ with 4 elements, and a singleton packet to the inner form $\mathrm{SL}_1(D)$. None of these packets contains the Steinberg representation of $\mathrm{SL}_2(K)$ and so each Π_φ is a supercuspidal L-packet with 4 elements.

Explicitly, ϕ assigns to $\mathrm{SL}_2(K)$ the supercuspidal packet

$$\{\pi(\phi,\rho_1), \pi(\phi,\rho_2), \pi(\phi,\rho_3), \pi(\phi,\rho_4)\}$$

and to $\mathrm{SL}_1(D)$ the singleton packet

$$\{\pi(\phi,\rho_0)\}$$

and this phenomenon occurs countably many times.

Theorem (Depth of ϕ_L)

Let L/K be a biquadratic extension, let ϕ be the Langlands parameter (2), $\phi = \alpha \circ \pi_{L/K}$ with $\alpha : \mathrm{Gal}(L/K) \to \mathrm{SO}_3(\mathbb{R})$.
(1) If t is the highest break in the upper ramification of $\mathrm{Gal}(L/K)$ then $d(\phi) = t$. The allowed values of $d(\phi)$ are $1, 3, 5, 7, \ldots$ except in case of two ramification breaks, when the allowed values are $3, 5, 7, \ldots$.
(2) For every $\pi \in \Pi_\phi(\mathrm{SL}_2(K))$ and $\pi \in \Pi_\phi(\mathrm{SL}_1(D))$ these integers provide lower bounds:

$$d(\pi) \geq d(\phi).$$

L-packets of cardinality 2

Let $\mathbf{W}_K$ be the Weil group of K and define

$$\mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to K^\times$$

to be the projection $(g, M) \mapsto g$ followed by the Artin reciprocity map

$$\mathbf{a}_K : \mathbf{W}_K \to K^\times.$$

Let E/K be a quadratic extension and let χ_E be the associated quadratic character of $K^\times$.
Let $[M]$ denote the image in $\mathrm{PSL}_2(\mathbb{C})$ of the element $M \in \mathrm{SL}_2(\mathbb{C})$. Consider the map

$$K^\times \to \mathrm{PSL}_2(\mathbb{C}), \qquad \alpha \mapsto \begin{bmatrix} \chi_E(\alpha) & 0 \\ 0 & 1 \end{bmatrix}$$

The composite map

$$\phi_E : \mathbf{W}_K \times \mathrm{SL}_2(\mathbb{C}) \to K^\times \to \mathrm{PSL}_2(\mathbb{C})$$

is then an L-parameter attached to χ_E. For the centralizer of the image, we have

$$C_{\mathrm{PSL}_2(\mathbb{C})}(\mathrm{im}\,\phi_E) = \{1, w\}$$

where w generates the Weyl group of the dual group $\mathrm{PSL}_2(\mathbb{C})$. Since there are two characters $1, \epsilon$ of $W = \{1, w\}$, there are two enhanced parameters $(\phi_E, 1)$ and (ϕ_E, ϵ), which parametrize the two elements in the L-packet Π_{ϕ_E}. We will write

$$\Pi_{\phi_E} = \{\pi_E^1, \pi_E^2\}. \qquad (4)$$

If $\gamma \in K^s$ is a root of $X^2 - X - \beta \in K[X]$, the quadratic extension $K(\gamma)$ is denoted also by $K(\wp^{-1}(\beta))$, with $\beta \in K$, where $\wp(X) = X^2 - X$. So the quadratic character

$$\chi_{n,j} = (-, u_j\varpi^{-2n-1} + \wp(K)]$$

is associated with the quadratic extension $E = K(\wp^{-1}(u_j\varpi^{-2n-1}))$.

There are two kinds of quadratic extensions: the unramified one $E_0 = K(\gamma_0)$ and countably many totally (and wildly) ramified $E = K(\gamma)$. The unramified quadratic extension has a single ramification break for $t = -1$.
Let E/K be a quadratic totally ramified extension. According to [Da], there is a single ramification break for $t = 2n+1$. Each value $2n+1$ occurs as a break, with $n \geq 0, 1, 2, 3, \ldots$.
Fix a basis $\mathcal{B} = \{u_1, \ldots, u_f\}$ of $\mathbb{F}_q/\mathbb{F}_2$ and let $u_j \in \mathcal{B}$. The next result shows how to realise the extension E/K.

Theorem (Depth of ϕ_E)

If $E = K(\wp^{-1}(u_j\varpi^{-2n-1}))$ then

$$d(\phi_E) = 2n + 1$$

with $n = 0, 1, 2, 3, 4, \ldots$.

References

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