Simple reduced L^p- operator crossed products with unique trace

Sanaz Pooya

PEPS-égalité "Correspondances de Langlands" - Projet INTEGER (GA no 266638)

Motivation to study L^p-operator crossed products

- Apart from many other interesting features of C*-crossed products, they are a rich source of producing new C*-algebras from old ones.
- The theory of crossed products of Banach algebras received less attention than of C*-algebras and von Neumann algebras.
- Working with an arbitrary Banach algebra is too general. But Banach algebras represented on L^p- spaces, for $p \in [1,\infty]$, are more convenient to work with.
  - Orthogonality is replaced by the complete classification of isometries on L^p-spaces for $p \neq 2$.
  - Positivity is remedied by good understanding of duality of L^p-spaces.
- In 2013, N. C. Phillips defined the notion of L^p-operator algebras as generalisation of C*-algebras in[5]. Some examples of such operator algebras are: L^p-Cuntz algebras, L^p-UHF algebras, **L^p-operator crossed products**

Some Definitions concerning L^p-operator algebras

Definition *Fix $p \in [1,\infty]$. An L^p-operator algebra is a Banach algebra A which is isometrically isomorphic to a norm closed subalgebra of $B(L^p(X,\mu))$ for some measure space $(X,\mathcal{B},\mu)$.*

Definition *Let A be a unital L^p-opertor algebra. A normalised trace on A is a linear functional satisfying the following conditions: $\tau(1) = \|\tau\| = 1$ and $\tau(ba) = \tau(ab)$ for all $a, b \in A$.*

Let $p \in [1,\infty)$, let G be a discrete group, and let (G, A, α) be an isometric L^p-dynamical system. Associate to any contractive representation $\pi\colon A \to B(L^p(X,\mu))$ the regular representation $\pi \rtimes \lambda\colon C_c(G, A, \alpha) \to B(l^p(G, L^P(X,\mu))$, where $\lambda\colon G \to B(l^p(G))$ is the L^p-left regular representation of G.

Definition *Let $p \in [1,\infty)$, let G be a discrete group, and let (G, A, α) be an isometric L^p-dynamical system. For $x \in C_c(G, A, \alpha)$, let $\|x\|_{\mathrm{r}} := \sup \|\pi \rtimes \lambda(x)\|$, where π runs through contractive representations of A. The L^p-operator reduced crossed product $A \rtimes_{\mathrm{r,p}} G$ of (G, A, α) is defined as*

$$A \rtimes_{\mathrm{r,p}} G := \overline{C_c(G, A, \alpha)}^{\|\cdot\|_{\mathrm{r}}}.$$

For $A = \mathbb{C}$, we obtain the notion of L^p-operator group algebras.

Powers averaging

Definition *A group G is a* Powers group *if for any non-empty finite subset $F \subset G \setminus \{e\}$ and any integer $k \geq 1$, there exist a partition $G = C \amalg D$ and elements $g_1, \dots, g_k \in G$ such that*

1. *$fC \cap C = \varnothing$ for all $f \in F$,*
2. *$g_1 D, \cdots, g_k D$ are pairwise disjoint.*

Example *Any non-elementary hyperbolic group without finite normal subgroup is a Powers group. In particular non abelian free groups, F_n, are Powers groups.*

Here is the argument that de la Harpe extracted from Powers' proof [6]. Until recently this averaging method formed the basis for most follow up results on simplicity and the uniqueness of trace for discrete groups.

We briefly recall how to estimate norms when averaging over the conjugation action of G on $C_c(G)$. Let $x \in C_c(G)$ with *support* $x =: F \subset G \setminus \{e\}$. Write $x = \sum_{f\in F} x_f u_f$. For F and $k \in \mathbb{N}$, find $g_1, \cdots, g_k$ and a partition $G = C \amalg D$ as in the definition of Powers groups. For $i \in 1, \cdots, k$, define projections $e_i : l^2(G) \to l^2(g_i D)$.

- $(1 - e_i) u_{g_i f g_i^{-1}} (1 - e_i) = 0$, by property (1)
- $\frac{1}{k}\|\sum_{i=1}^{k} u_{g_i} x u_{g_i^{-1}}\| \leq \frac{1}{k}(\|\sum_{i=1}^{k} e_i u_{g_i} x u_{g_i^{-1}}\| + \|\sum_{i=1}^{k}(1-e_i) u_{g_i} x u_{g_i^{-1}} e_i)\| \leq \frac{2\|x\|}{\sqrt{k}}$. by property (2)
  Therefore large k makes the averaging norm tend to 0.

We generalised this argument to the L^p-case and profit from it in our results.

Connection to other's work

- In 1975, Powers announced that $C^*_{\mathrm{r}}(F_2)$ is simple and has a unique trace. This was the first example of a group with these properties.
- In 1985, de la Harpe formalised Powers' idea and introduced Powers groups. Reduced group C*-algebras of Powers groups are simple and have a unique trace.
- de la Harpe and Skandalis proved in [3]:
  If G is a Powers group and if A is a unital G-simple C*-algebra. Then $A \rtimes_{\mathrm{r}} G$ is simple and all traces factor through the conditional expectation E on A, where
  $$E : A \rtimes_{\mathrm{r}} G \to A : au_g \mapsto a\delta_{\mathrm{g,e}}$$
  .
- In 2014, the astonishing work of Kalantar, Kennedy [2] and Breuillard, Kalantar, Kennedy, Ozawa [1] clarified exactly which discrete groups have simple reduced group C*-algebras and unique trace. In fact, $C^*_r(G)$ is simple if and only if there is a boundary of G (in a technical sense) on which the action of G is essentially free.

Main results

Theorem 1 *Let $p \in (1,\infty)$, let G be a Powers group, and let (G, A, α) be an isometric L^p-dynamical system. Then each normalised trace τ on $A \rtimes_{\mathrm{r,p}} G$ satisfies $\tau = \tau \circ E$, where E is the conditional expectation on A. In particular, if A has a unique trace then so does $A \rtimes_{\mathrm{r,p}} G$.*

Sketch of Proof *By means of Powers averaging we see that for a given ϵ and $f \in G \setminus \{e\}$, we have*

$$|\tau(u_f)| = |\tau(\frac{1}{k}\sum_{i=1}^{k} u_{g_i} u_f u_{g_i^{-1}})| \leq \frac{1}{k}\|\sum_{i=1}^{k} u_{g_i} u_f u_{g_i^{-1}}\| \leq \epsilon$$

Hence for such elements, $\tau(u_f) = 0$. Therefore normalised traces concentrate on A ($\cong Au_e$).

Theorem 2 *Let $p \in (1,\infty)$ and let G be a Powers group. If A is G-simple and (G, A, α) is an isometric L^p-dynamical system, then $A \rtimes_{\mathrm{r,p}} G$ is simple.*

Sketch of Proof *We show that a non trivial ideal in $A \rtimes_{\mathrm{r,p}} G$ contains an invertible element. Indeed, by applying Powers averaging to $x - 1_A$ for some specific element x in the ideal, we find that x is invertible.*

For $p = 2$, we reobtain the classical results:

- $p = 2$, $G = F_2$ and $A = \mathbb{C}$ reduces to Powers' result. [6]
- $p = 2$, $A =$ unital G-simple C*-algebra reduces to de la Harpe and Skandalis' result. [3]

Note that we excluded the case $p = 1$ from our results since

- $p = 1$ and $G =$ discrete group, the L^1-operator group algebra of G is just $l^1(G)$ which is not simple.

Outlook

In recent work of Kalantar, Kennedy [2] and Breuillard, Kalantar, Kennedy, Ozawa [1], there has been a considerable advance in understanding simplicity and uniqueness of the trace for reduced C*-algebras of discrete groups by studying their action on Furstenberg boundary. These results rely on the notion of positivity. So it is interesting to know in how far one can transfer them to the L^p-case.

References

1. E. Breuillard, M. Kalantar, M. Kennedy, N. Ozawa, C*-simplicity and the unique trace property for discrete groups, arXiv: 1410.2518
2. M. Kalantar, M, Kennedy, Boundaries of reduced C*-algebras of discret groups, arXiv: 1405.4359
3. P. de la Harpe, G. Skandalis, Powers' property and simple C*-algebras, Mathematische Annalen, 273 (1986), 241–250.
4. Sh. Hejazian, S. Pooya, Simple reduced L^p operator crossed products with the unique trace, accepted for publication in Journal of Operator theory
5. N. C. Phillips, Crossed products of L^p-operator algebras and the K-theory of Cuntz algebras on L^p-spaces, arXiv: 1309.6406
6. R. T. Powers, Simplicity of the C^*-algebra associated with the free group on two generators, Duke J. Math. 42 (1975), 151–156.

2015 Université de Neuchâtel
sanaz.pooya@unine.ch