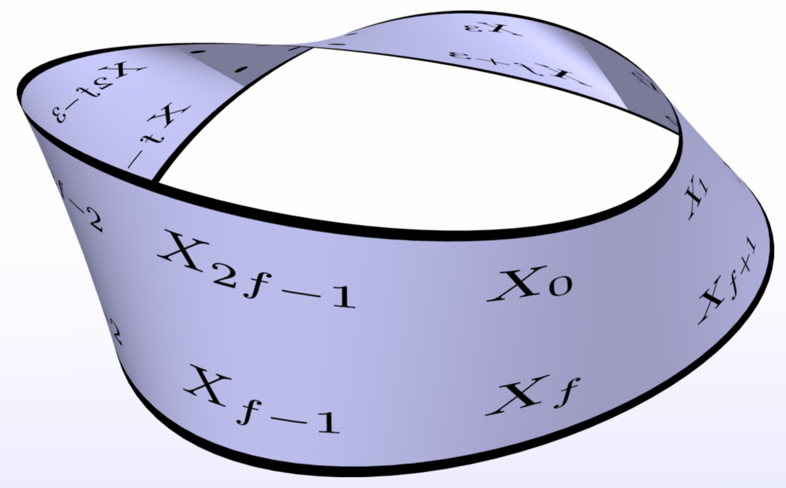
Genetics of Galois deformations

Xavier Caruso, Agnès David, Ariane Mézard
IRMAR (Rennes), LMB (Besançon), IMJ (Paris)



Deformation data

- p prime at least 5, f integer (at least 2)
- $E/\mathbb{Q}_p$ finite, large enough, $\mathcal{O}_E$, ϖ_E, k_E
- ω_f, ω_{2f} fundamental characters of level f, $2f$

Galois representation

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f}) \longrightarrow \mathrm{GL}_2(k_E) \text{ irreducible}$$

$$\overline{\rho} = \mathrm{Ind}_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^{2f}})}^{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p^f})} (\omega_{2f}^h) \text{ for } h \text{ in } \mathbb{N}$$

Inertial type $\mathrm{t} = [\omega_f]^\gamma \oplus [\omega_f]^{\gamma'}$ with $\gamma \neq \gamma'$

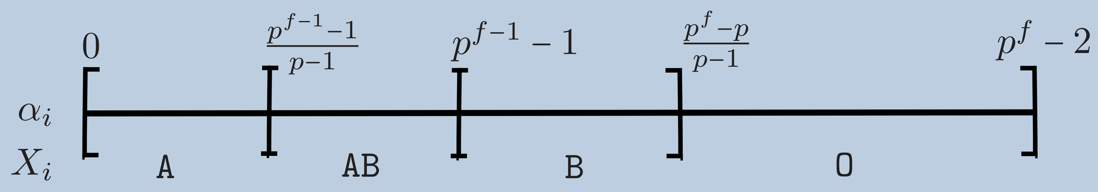
$$\mathrm{t} : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{ur}}) \longrightarrow \mathrm{GL}_2(E)$$

Serre weights $\mathcal{D}(\overline{\rho})$, $\mathcal{D}(\mathrm{t})$ given by LLC

Gene of $\overline{\rho}$ and t

For all i in $\mathbb{N}$, we define α_i in $[\![0; p^f-2]\!]$ by

$$\alpha_i \equiv \left\lfloor \frac{p^i h}{p^f+1} \right\rfloor - p^i \gamma' \mod (p^f-1).$$



X_0	X_1	X_2	$\cdots$	X_{f-2}	X_{f-1}
X_f	X_{f+1}	X_{f+2}	$\cdots$	X_{2f-2}	X_{2f-1}

We endow the gene with diagonal and horizontal bonds, depending on which letter (A or B) is dominant in each of the f columns.

Deformation ring and Kisin variety

- $R_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$ parametrises potentially Barsotti–Tate deformations of $\overline{\rho}$ with Galois type t.

Breuil–Mézard Conjecture (Kisin's version). *The Hilbert–Samuel multiplicity of the quotient $R_{\mathrm{BT}}(\overline{\rho},\mathrm{t})/\varpi_E R_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$ equals the number $\mu(\overline{\rho},\mathrm{t})$ of Serre weights in $\mathcal{D}(\overline{\rho}) \cap \mathcal{D}(\mathrm{t})$.*

- $\mathscr{GR}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$ parametrises some Breuil–Kisin lattices of height 1 with descent data t.
  It is a projective scheme over $R_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$, with same generic fiber.
  $\overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t}) := \mathscr{GR}_{\mathrm{BT}}(\overline{\rho},\mathrm{t}) \times_{\mathrm{Spec}(R_{\mathrm{BT}}(\overline{\rho},\mathrm{t}))} \mathrm{Spec}(k_E)$ special fiber

Geometry of $\overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$

Theorem.

(i) The special fiber $\overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$ naturally appears as a reduced closed subscheme of $(\mathbb{P}^1_{k_E})^f$.

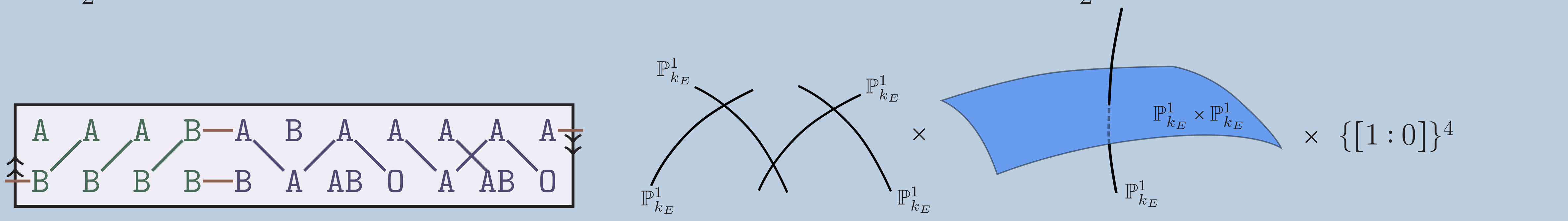
(ii) The irreducible components of $\overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$ are products of projective lines.

(iii) The special fiber $\overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$ is connected.

The gene and its diagonal bonds give explicit equations and description of irreducible components.

Non-equidimensional example for $f = 11$

$$h \equiv \tfrac{p+1}{2} + (p-1)p^3 + (p-1)p^5 + p^8 + p^9 + p^{10} \mod (p^{11}+1) \; ; \; \gamma - \gamma' \equiv -\tfrac{p+3}{2} - p^2 - 2p^3 + p^7 \mod (p^{11}-1)$$



$$\overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t}) = \left\{ \begin{array}{l|l} ([x_i : x_{11+i}])_{i\in[\![0;10]\!]} \in (\mathbb{P}^1_{k_E})^{11} & x_{18} = x_{21} = 0, [x_8 : x_{19}] = [x_9 : x_{20}], \\ & x_1x_{11} = x_2x_{12} = x_3x_{13} = x_4x_{16} = x_{16}x_6 = x_6x_{18} = x_7x_{19} = x_9x_{21} = 0 \end{array} \right\}$$

Stratification by the shape

The *shape* (or *genre*) of a point of $\overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$ is a data in $\{\mathrm{I},\mathrm{II}\}^f$ describing the action of the frobenius in the corresponding Breuil–Kisin module. It is encoded by the gene with its diagonal and horizontal bonds.

Proposition. *The shape defines a stratification on $\overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$ by reduced locally closed subschemes.*

Gene	$\overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$	$\mu(\overline{\rho},\mathrm{t})$	$R_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$
B AB / 0 0	I×II	2	$\mathcal{O}_E[[X,Y,T]]/(XY+p)$
A A / AB 0	II×I	2	$\mathcal{O}_E[[X,Y,T]]/(XY+p)$
B A / B B	I×II	2	? ?
A B / AB 0	II×I, I×II, I×I, $\mathbb{P}^1$	2	$\mathcal{O}_E[[X,Y,T]]/(XY+p^2)$

Gene	$\overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$	$\mu(\overline{\rho},\mathrm{t})$
A B B / A A A	I×II×I, I×I×II, I×I×I, $\mathbb{P}^1$	2
0 B B / A B AB	II×I×I, I×I×II, I×I×I, $\mathbb{P}^1$	3
0 A B / AB 0 AB	II×II×II, II×I×I, $\mathbb{P}^1$	6

Gene	$\overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$	$\mu(\overline{\rho},\mathrm{t})$
0 B B / A—A AB	I×II×I, I×I×I, I×I×I, $\mathbb{P}^1$, I×I×II, II×I×I, $\mathbb{P}^1$	2
0 A B / A—B AB	II×II×II, II×I×I, I×I×II, $\mathbb{P}^1$, $\mathbb{P}^1$	4

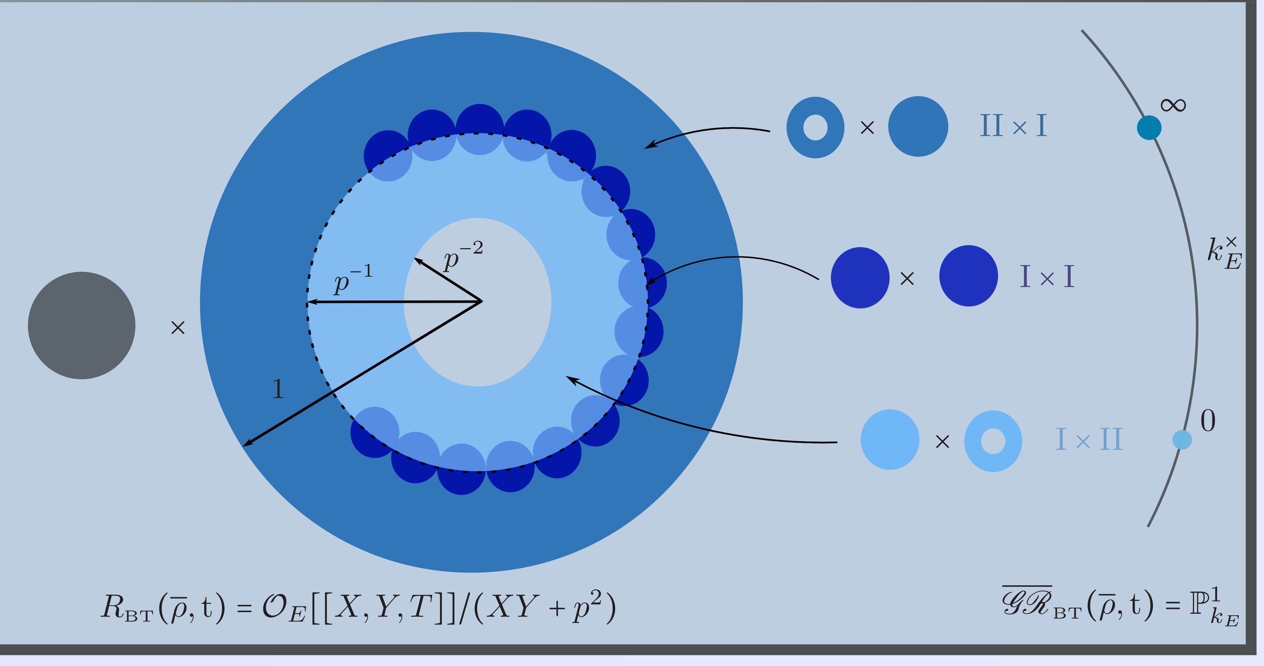
Gene	$\overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$	$\mu(\overline{\rho},\mathrm{t})$
A—A B / A—B B	$\mathbb{P}^1\times\mathbb{P}^1$, II×I×I, II×II×I, I×II×I, I×I×II, I×II×II, $\mathbb{P}^1\times\mathbb{P}^1$	4
0 B—A / A—B—AB	II×II×I, I×I×I, I×II×I, I×II×II, II×I×I, I×I×II, II×I×II, $\mathbb{P}^1\times\mathbb{P}^1$	5

Application. *The Breuil–Mézard Conjecture is true for $f = 2$ and non degenerate $\overline{\rho}$.*

The stratification by the shape also provides a guide for a construction, by blowing-up, of a good candidate for $R_{\mathrm{BT}}(\overline{\rho},\mathrm{t})[1/p]$.

Gluing rigid tubes

- $D_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$ rigid space of $R_{\mathrm{BT}}(\overline{\rho},\mathrm{t})[1/p]$
- **Specialisation map**
  $$\mathrm{sp} : D_{\mathrm{BT}}(\overline{\rho},\mathrm{t}) \longrightarrow \overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$$
  On closed points, it corresponds to the reduction modulo ϖ_E of Breuil–Kisin modules.
- $\forall \xi \in \overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$, $\mathrm{sp}^{-1}(\xi)$ is a product of open discs (I) and annulus (II) given by the shape.
- $D_{\mathrm{BT}}(\overline{\rho},\mathrm{t}) = \displaystyle\coprod_{\xi \in \overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})} \mathrm{sp}^{-1}(\xi)$



Questions and conjectures

- Use connexity of $\overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$ to prove irreducibility of $R_{\mathrm{BT}}(\overline{\rho},\mathrm{t})[1/p]$.
- Is $R_{\mathrm{BT}}(\overline{\rho},\mathrm{t})/(\varpi_E R_{\mathrm{BT}}(\overline{\rho},\mathrm{t}))$ reduced ? Is $R_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$ normal ?

Conjectures. *For non degenerate inertial type* t,

⋆ *the generic fiber $R_{\mathrm{BT}}(\overline{\rho},\mathrm{t})[1/p]$ is determined by $\overline{\mathscr{GR}}_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$ equipped with its stratification,*

⋆ *the ring $R_{\mathrm{BT}}(\overline{\rho},\mathrm{t})$ is determined by the gene of $(\overline{\rho},\mathrm{t})$.*

References

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Algorithms

http://cethop.math.cnrs.fr/

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