Bernstein center of supercuspidal blocks

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Abstract

Let $\mathbf{G}$ be a tamely ramified connected reductive group defined over a non-archimedean local field k. We show that the Bernstein center of a tame supercuspidal Bernstein block of $\mathbf{G}(k)$ is isomorphic to the Bernstein center of a depth zero supercuspidal block of a twisted Levi subgroup of $\mathbf{G}(k)$. As a consequence, we obtain that the centers of the Hecke algebras in Yu's conjecture are isomorphic. We also give a Galois side analogue of the finiteness condition satisfied by supercuspidal blocks.

Introduction

Let $\mathbf{G}$ be a connected reductive group defined over a non archimedean local field k. Assume that $\mathbf{G}$ splits over a tamely ramified extension k^t of k. We will denote the group of k-rational points of $\mathbf{G}$ by G and likewise for other algebraic groups. In [5], Jiu-Kang Yu gives a very general constuction of a class of supercuspidal representations of G which he calls *tame*. A tame supercuspidal representation $\pi = \pi_\Sigma$ of G is constructed out of a depth zero supercuspidal representation π_0 of G^0 and some additional data, where $\mathbf{G}^0$ is a *twisted* Levi subgroup of $\mathbf{G}$. By twisted, we mean that $\mathbf{G}^0 \otimes k^t$ is a Levi factor of a parabolic subgroup of $\mathbf{G} \otimes k^t$. The additional data, together with $\mathbf{G}^0$ and π_0 is what we are denoting by Σ in the notation π_Σ. In [2], Kim showed that under certain hypothesis, which are met for instance when the residue characteristic is large, these tame supercuspidals exhaust all the supercuspidals of G.

The depth zero supercuspidal π_0 of G^0 is compactly induced from (K^0, ϱ_0) where K^0 is a compact mod center open subgroup of G^0 and ϱ_0 is a representation of K^0. The constructed representation π_Σ is compactly induced from (K, ϱ), where K is a compact mod center open subgroup of G containing K^0 and ϱ is a representation of K. The representation ϱ is of the form $\varrho_0 \otimes \kappa$, where ϱ_0 is seen as a representation of K by extending from K^0 trivially and κ is a representation of K constructed out of the part of Σ which is independent of ϱ_0.

Let $\mathfrak{Z}^\pi$ (resp. $\mathfrak{Z}_0^{\pi_0}$) denote the *Bernstein center* of the *Bernstein block* (see Section 3 for these terms) of G (resp. G^0) containing π (resp. π_0). We show that

Theorem. $\mathfrak{Z}^\pi \cong \mathfrak{Z}_0^{\pi_0}$. *Thus, the Bernstein center of a tame supercuspidal block of G is isomorphic to the Bernstein center of a depth zero supercuspidal block of a twisted Levi subgroup of G.*

Let $\mathcal{H}(G, \varrho)$ (resp. $\mathcal{H}(G^0, \varrho_0)$) denote the Hecke algebra of the type constucted out of (K, ϱ) (resp. (K^0, ϱ_0)) (see [1, Sec. 5.4]). As a consequence of the above theorem, we obtain

$$Z(\mathcal{H}(G, \varrho)) \cong Z(\mathcal{H}(G^0, \varrho_0)).$$

For a compact open subgroup J of G, a well known theorem of Bernstein states that there exist only finitely many supercuspidal Bernstein blocks having a J-spherical representation. In Theorem 7, we give a Galois side analogue of this result.

1 Notations

Throughout this poster, k denotes a non-archimedean local field. For an algebraic group $\mathbf{G}$ defined over k, we will denote its k-rational points by G. Center of $\mathbf{G}$ will be denoted by $\mathbf{Z_G}$. The category of smooth representations of G will be denoted by $\mathfrak{R}(G)$. Let W_k denote the Weil group of k and I_k denote its inertia subgroup. We fix a Frobenius element Fr in W_k.

2 Yu's construction [5]

Let $\mathbf{G}$ be a connected reductive group defined over a non-archimedean local field k. A twisted k-Levi subgroup $\mathbf{G}'$ of $\mathbf{G}$ is a reductive k-subgroup such that $\mathbf{G}' \otimes_k \bar{k}$ is a Levi subgroup of $\mathbf{G} \otimes_k \bar{k}$. Yu's construction involves the notion of a generic $\mathbf{G}$-datum. It is a quintuple $\Sigma = (\vec{\mathbf{G}}, y, \vec{r}, \vec{\phi}, \rho)$ satisfying the following:

1. $\vec{\mathbf{G}} = (\mathbf{G}^0 \subsetneq \mathbf{G}^1 \subsetneq \ldots \subsetneq \mathbf{G}^d = \mathbf{G})$ is a tamely ramified twisted Levi sequence such that $\mathbf{Z}_{\mathbf{G}^0}/\mathbf{Z}_G$ is anisotropic.
2. y is a point in the extended Bruhat-Tits building of $\mathbf{G}^0$ over k.
3. $\vec{r} = (r_0, r_1, \cdots, r_{d-1}, r_d)$ is a sequence of positive real numbers with $0 < r_0 < \cdots < r_{d-2} < r_{d-1} \leq r_d$ if $d > 0$, $0 \leq r_0$ if $d = 0$.
4. $\vec{\phi} = (\phi_0, \cdots, \phi_d)$ is a sequence of quasi-characters, where ϕ_i is a G^{i+1}-*generic quasi-character* [5, Sec. 9] of G^i; ϕ_i is trivial on G^i_{y,r_i+}, but nontrivial on G^i_{y,r_i} for $0 \leq i \leq d-1$. If $r_{d-1} < r_d$, ϕ_d is nontrivial on G^i_{y,r_d} and trivial on G^d_{y,r_d+}. Otherwise, $\phi_d = 1$. Here $G^i_{y,\cdot}$ denote the filtration subgroups of the parahoric at y defined by Moy-Prasad (see [4]).
5. ρ is an irreducible representation of $G^0_{[y]}$, the stabilizer in G^0 of the image $[y]$ of y in the reduced building of $\mathbf{G}^0$, such that $\rho|G^0_{y,0+}$ is isotrivial and $c\text{-}\mathrm{Ind}_{G^0_{[y]}}^{G^0}\rho$ is irreducible and supercuspidal.

Let $K^0 = G^0_{[y]}$ and $K^i = G^0_{[y]}G^1_{y,s_0}\cdots G^i_{y,s_{i-1}}$ where $s_j = r_j/2$ for $i = 1, \ldots, d$. In [5, Sec. 11], Yu constructs certain representation κ of K^d which is independent of ρ and constructed only out of $(\vec{\mathbf{G}}, y, \vec{r}, \vec{\phi})$. Extend ρ trivially to a representation of K^d and write $\rho_\Sigma := \rho \otimes \kappa$.

Theorem 1 (Yu)**.** $\pi_\Sigma := c\text{-}\mathrm{Ind}_{K^d}^G \rho_\Sigma$ *is irreducible and thus supercuspidal.*

3 Bernstein center and Bernstein decomposition

In [3, Section 7], Kottwitz defined a functorial homomorphism $\kappa'_G : G \twoheadrightarrow X_*(\mathbf{Z_G})^{\mathrm{Fr}}_{I_k}$. The map κ'_G induces a functorial surjective map:

$$\kappa_G : G \twoheadrightarrow X_*(\mathbf{Z_G})^{\mathrm{Fr}}_{I_k}/\,\mathrm{torsion}\,. \tag{1}$$

Let ${}^\circ G := \ker(\kappa_G)$ and let $X_{\mathrm{nr}}(G) := \mathrm{Hom}(G/{}^\circ G, \mathbb{C}^\times)$ denote the group of *unramified characters* of G.

Consider the collection of all cuspidal pairs (L, σ) consisting of a Levi subgroup L of G and an irreducible cuspidal representation σ of L. Define an equivalence relation $\sim$ on the class of all cuspidal pairs by

$$(L, \sigma) \sim (M, \tau) \text{ if } {}^gL = M \text{ and } {}^g\sigma \cong \tau\nu,$$

for some $g \in G$ and some $\nu \in X_{\mathrm{nr}}(M)$. Write $[L, \sigma]$ for the equivalence class of (L, σ) and $\mathfrak{B}(G)$ for the set of all equivalence classes. The set $\mathfrak{B}(G)$ is called the *Bernstein spectrum* of G. We say that a smooth irreducible representation π has *inertial support* $\mathfrak{s} := [L, \sigma]$ if π appears as a subquotient of a representation parabolically induced from some element of $\mathfrak{s}$. Define a full subcategory $\mathfrak{R}^{\mathfrak{s}}(G)$ of $\mathfrak{R}(G)$ as follows: a smooth representation π belongs to $\mathfrak{R}^{\mathfrak{s}}(G)$ iff each irreducible subquotient of π has inertial support $\mathfrak{s}$.

Theorem 2 (Bernstein)**.** *We have*

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

Definition 3. The endomorphism ring of the identity functor of $\mathfrak{R}(G)$ (resp. $\mathfrak{R}(G)^{\mathfrak{s}}$) is called the *Bernstein center* of $\mathfrak{R}(G)$ (resp. $\mathfrak{R}(G)^{\mathfrak{s}}$).

4 Centers of supercuspidal blocks

We use the notations of Section 2. So $\mathbf{G}$ is a connected reductive group over k, $\Sigma = (\vec{\mathbf{G}}, y, \vec{r}, \vec{\phi}, \rho)$ is a generic $\mathbf{G}$-datum, $K^0 = G^0_{[y]}$ and $K^i = G^0_{[y]}G^1_{y,s_0}\cdots G^i_{y,s_{i-1}}$ where $s_j = r_j/2$ for $i = 1, \ldots, d$. Then in [5], Yu constructs a representation ρ_Σ of K^d such that $\pi_\Sigma := c\text{-}\mathrm{Ind}_{K^d}^G \rho_\Sigma$ is irreducible and thus supercuspidal. The representation $\pi_0 = c\text{-}\mathrm{Ind}_{K^0}^{G^0}\rho$ is depth zero supercuspidal. Write ${}^\circ K^d := K^d \cap {}^\circ G$ (resp. ${}^\circ K^0 := K^0 \cap {}^\circ G^0$) and ${}^\circ\rho_\Sigma := \rho_\Sigma|{}^\circ K^d$ (resp. ${}^\circ\rho = \rho|{}^\circ K^0$). Here ${}^\circ G$ is as defined in Section 3. Then $({}^\circ K^d, {}^\circ\rho_\Sigma)$ (resp. $({}^\circ K, {}^\circ\rho)$) is an $\mathfrak{s} := [G, \pi_\Sigma]_G$ (resp. $\mathfrak{s}_0 := [G^0, \pi_0]_{G^0}$) type [5, Corr. 15.3].

Let $\mathfrak{Z}(G)$ (resp. $\mathfrak{Z}(G)^{\mathfrak{s}}$, resp. $\mathfrak{Z}(G^0)^{\mathfrak{s}_0}$) be the Bernstein center of the category $\mathfrak{R}(G)$ (resp. $\mathfrak{R}(G)^{\mathfrak{s}}$, resp. $\mathfrak{R}(G^0)^{\mathfrak{s}_0}$).

Theorem 4. *The Bernstein center of a tame supercuspidal block of G is isomorphic to the Bernstein center of a depth zero supercuspidal block of a twisted Levi subgroup of G. More precisely, $\mathfrak{Z}(G)^{\mathfrak{s}} \cong \mathfrak{Z}(G^0)^{\mathfrak{s}_0}$.*

For an algebra $\mathcal{A}$, denote by $Z(\mathcal{A})$ the center of $\mathcal{A}$. Let $\mathcal{H}(G, \rho_\Sigma)$ (resp. $\mathcal{H}(G^0, \rho)$) denote the Hecke algebra of the type $({}^\circ K^d, {}^\circ\rho_\Sigma)$ (resp. $({}^\circ K^0, {}^\circ\rho)$).

Corollary 5. $Z(\mathcal{H}(G, \rho_\Sigma)) \cong Z(\mathcal{H}(G^0, \rho))$.

Now suppose that π_Σ satisfied the conditions (5.5) of [1]. These are satisfied for instance when $\mathbf{G} = \mathrm{GL}(n, k)$. In that case, $\mathcal{H}(G, \rho_\Sigma)$ is commutative. In this situation we have:

Corollary 6. $\mathcal{H}(G, \rho_\Sigma) \cong \mathcal{H}(G^0, \rho)$.

5 A result on the parameters attached to supercuspidal blocks

Let $\Phi(G)$ denote the set of Langlands parameters of G. We have a well defined action

$$\mathrm{H}^1(W_k, Z(\hat{\mathbf{G}})) \times \Phi(G) \to \Phi(G), \qquad [\alpha]\cdot[\phi] \mapsto [\alpha \cdot \phi], \tag{2}$$

where $Z(\hat{\mathbf{G}})$ is the center of the complex dual $\hat{\mathbf{G}}$ of $\mathbf{G}$

Kottwitz homomorphism allows one to get a canonical isomorphism $X_{\mathrm{nr}}(G) \cong \mathrm{H}^1(W_k/I_k, (Z(\hat{\mathbf{G}})^{I_k})^\circ)$. This gives an action of $X_{\mathrm{nr}}(G)$ on $\Phi(G)$.

Theorem 7. *Let $\mathbf{G}$ be a connected reductive group over k. Then up to $X_{\mathrm{nr}}(G)$-action, there exist only finitely many discrete Langlands parameters for G which are trivial on a given compact open normal subgroup of W_k.*

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